The King's School Canterbury

13+ Scholarship Examinations 2019

MATHEMATICS II

1 hour (including five minutes suggested reading time)

Use the reading time wisely; gain an overview of the paper and start to think of how you will answer the questions.

Do as many questions as you can (clearly numbered) on the lined paper provided. Clearly name each sheet used.

The questions are not of equal length or mark allocation. Move on quickly if stuck. ***You are not expected to finish everything.***

You are expected to use a calculator where appropriate, but you must show ***full and clear working****, diagrams and arguments wherever you can. Marks will be awarded for method as well as answers: merely writing down an answer might score very few marks.*

Complete solutions are preferable to fragments.

This paper has eight questions.

1 How many whole numbers between 1 and 2019 inclusive are **not** divisible by 3 or 7?

2 Theresa and Boris play a game in which there are only winners and losers i.e. no draws.

Each of them is equally likely to win each game.

The first player to win four games becomes champion and then the game stops.

Theresa wins the first two games. What is the probability that Boris ends up as champion?

3 Lamech is as old as Methuselah was when Lamech was as old as Methuselah had been when Lamech was half as old as Methuselah is.

Their present ages sum to 1650.

How old is Methuselah now?

4 I am marking twelve scholarship papers and decide to give them grades at random, with an equal number of grades A, B, C and Z.

(a) Explain carefully why there are 220 different ways of allocating the A grades.

(b) Work out the total number of ways of assigning the grades A, B , C, and Z as described.

5 In this question you are given that a Pythagorean triple is a set of three whole numbers *a,b,c* such that

$$a^2 + b^2 = c^2$$

(a) Explain why an odd number may be written in general as $2n - 1$.

(b) Professor Atiyah says: "If you square an odd number, then halve it and use the whole numbers either side of the answer you have a Pythagorean triple".

Use algebra to prove carefully that this statement is true.

[e.g. $9^2 = 81$, $81/2 = 40.5$, and then (9,40,41) is a Pythagorean triple, with $9^2 + 40^2 = 41^2$.]

6 In this question you can use the following definitions:

If A and B are two numbers, their three main means are

Arithmetic	Geometric	Harmonic
$\frac{A+B}{2}$	$\sqrt{AB}$	$\frac{2}{\frac{1}{A}+\frac{1}{B}}$

You can also use the fact that, if A ≠ B then Arithmetic mean > Geometric mean > Harmonic mean .

In the First Century AD Hero of Alexandria devised a way of finding approximate values for square roots e.g. √3

Consider two numbers a and $b = 3/a$.

(a) What is their geometric mean?

(b) Write down an expression for the arithmetic mean.

(c) Write an expression for the harmonic mean and simplify as much as you can.

(d) Combine these to form an inequality.

(e) Take a starting value of $a = 5/3$, and using the inequalities derived above, show that

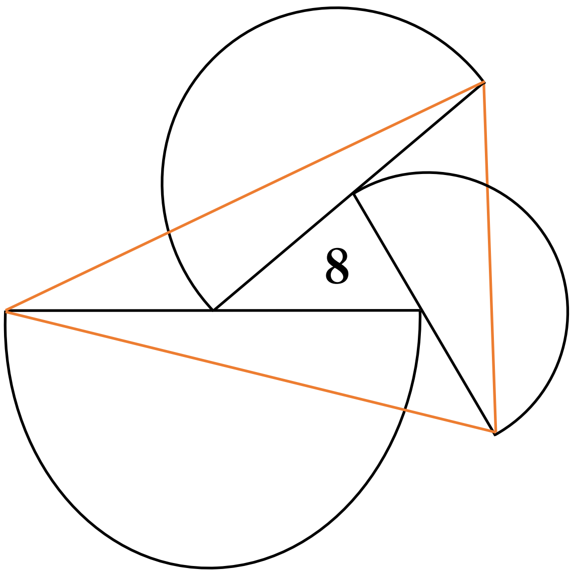
$$\frac{45}{26} < \sqrt{3} < \frac{26}{15}$$

and then taking $a = \frac{26}{15}$ show that

$$\frac{2340}{1351} < \sqrt{3} < \frac{1351}{780}$$

7 In the diagram the vertices of the smaller triangle (of area 8cm^2) are also the centres of the semicircles.

What is the area of the larger triangle?



Not to scale

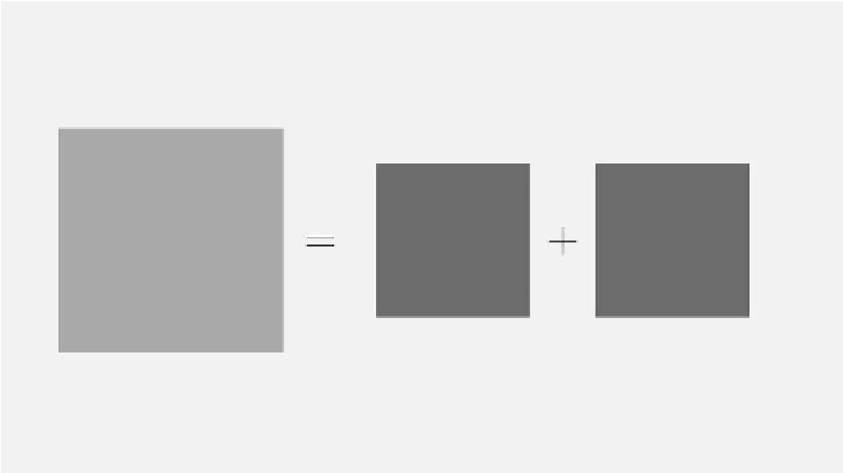
8 In this question we will investigate what sort of number $\sqrt{2}$ is.

Suppose $\sqrt{2}$ is a fraction, so we can write $\sqrt{2} = p/q$ for some **whole numbers** p and q.

(a) Rearrange this to write p^2 in terms of q^2 .

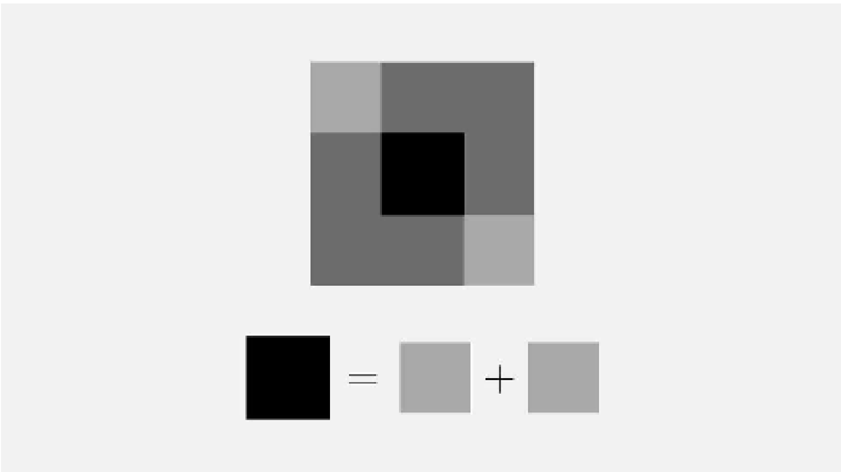
(b) If this fraction p/q is in **lowest terms** what can we say about p and q? Explain also why we can always cancel a fraction down to lowest terms.

(c) What is the connection between the picture of square areas below and parts (a) and (b)? What type of numbers are the side lengths of these squares?



(d) Placing the two smaller squares on the larger one, what must the sum of the areas of the corner squares equal?

(e) Why?



(f) Why are these smaller squares' side lengths all whole numbers?

(g) Now try to consider your answers to (d) to (f) with your answers to (a) to (c). What is the problem?

(h) What can you conclude?

END OF QUESTION PAPER

The King's School
Canterbury

13+ Scholarship Examinations

May 2021

MATHEMATICS II

FURTHER MATHEMATICS

1 hour **(including five minutes suggested reading time)**

Use the reading time wisely; gain an overview of the paper and start to think of how you will answer the questions.

Do as many questions as you can in the spaces provided. Use extra paper only if you have to, and make sure you clearly name any sheets used.

The questions are not of equal length or mark allocation. Move on quickly if stuck. You are not expected to finish everything.

You are expected to use a calculator where appropriate, but you must show full and clear working, diagrams and arguments wherever you can. Marks will be awarded for method as well as answers: merely writing down an answer might score very few marks.

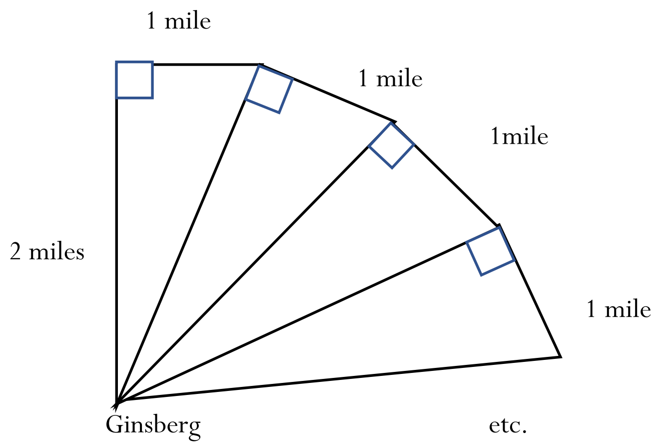
Complete solutions are preferable to fragments.

This paper has seven questions.

1

(a) Kerouac drives his car in the desert. He starts a certain distance *d* miles due West from Ginsberg and drives in a straight line for 45 miles until he is two miles due North of Ginsberg. What was the **exact** value of *d* ?

(b) Next, Kerouac drives one mile East. He turns so the car points at right angles to the line from him to Ginsberg, drives one mile and then repeats this turn-and-drive step another sixty times.



What is his **exact** distance from Ginsberg now?

2

In this question, use the following figures, and write down any other measurements you assume and use.

Diameter of the Earth: 12742 km Height of Mount Everest: 8849 m

A biologist friend tells me that human fingers can feel bumps as small as 0.00001 mm, and claims that, if the Earth were shrunk down to the size of a tennis ball, we could still feel Mount Everest.

Showing all your working, verify this claim or demonstrate that it is wrong.

3

Boris lies on Wednesdays, Fridays and Sundays and tells the truth the rest of the week.

Dom is so tired from sitting hard scholarship papers that he does not know what day it is.

Dom: what day is it today?

Boris: Sunday.

Dom: what day is it tomorrow?

Boris: Thursday.

What day is it today?

4

In another scholarship examination there is an option to do a paper on **hard sums**, or choose other papers.

Everyone choosing the hard sums paper has already done some hard sums before.

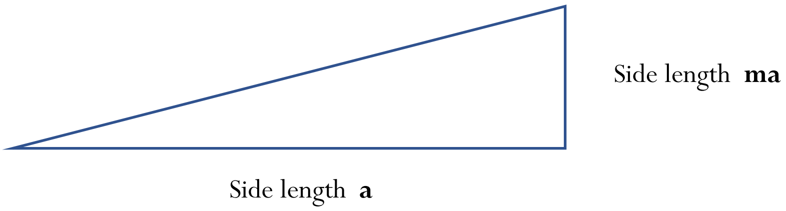
90% of those choosing other papers had never done hard sums before.

64% of all those making a choice have done hard sums before.

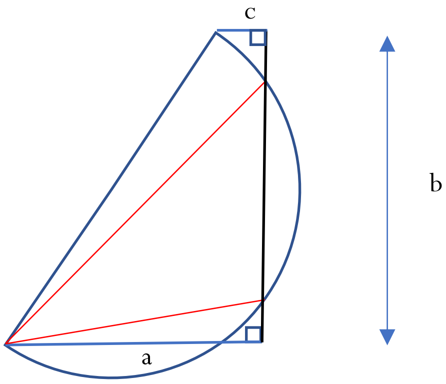
What percentage choose hard sums?

5

We define **gradient** (slope) as rise/run, so in a right-angled triangle you have this connection between the lengths of the two shorter sides when the hypotenuse has gradient m.



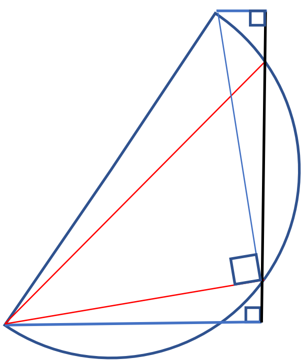
Consider the diagram below, with three perpendicular/parallel straight lines of length a, b, c, **joined end-to-end**, and a semicircle drawn with diameter as shown, joining the free ends of the lengths a and c.



Consider the lower of the two points where the circle cuts through the side of length b .

Draw a line from the upper end of the diameter to this point to make another triangle (see below).

You may use without proof the fact that the angle between these two triangles is 90°.



(a) How is this triangle related to the triangle with base side of length a? Explain your answer.

(b) If the hypotenuse of the **latter** triangle has gradient m, what is its height?

(c) Use your answers to (a) and (b) to find an expression for c in terms of a, b and m.

(d) Use your answer to (c) to show that the gradient m must satisfy the quadratic equation

$$am^2 - bm + c = 0$$

6

In this question consider a, b and c to be any non-negative numbers.

We define a new operation $\oplus$ as

$$a \oplus b = a + \lfloor b \rfloor$$

where $\lfloor b \rfloor$ is the integer (whole-number) part of b, so e.g. $\lfloor 2.7 \rfloor = 2$

For example, $3.7 \oplus 4.8 = 3.7 + 4 = 7.7$

(a) Explain carefully why, for all a,b,c

$$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$

You might give some examples to start with but try to make a general argument.

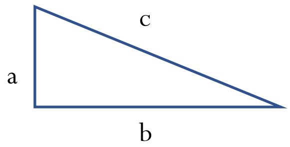
(b) Explain why, in general

$$a \oplus b \neq b \oplus a$$

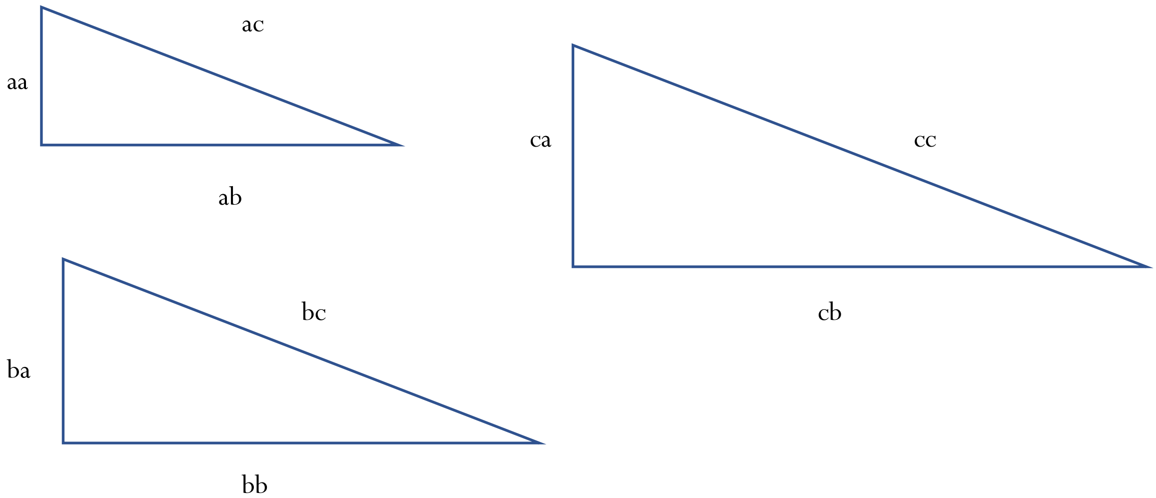
7

In this question we are going to prove the Pythagorean Theorem, which states that for a right-angled triangle of sides a,b,c (c being the hypotenuse) then

$$c^2 = a^2 + b^2$$



Take the triangle and make three enlarged copies of it like this:



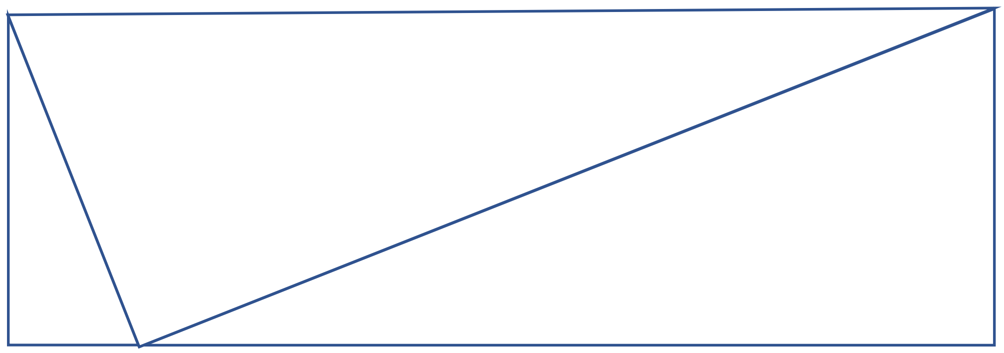
[These are enlargements by scale factors a, b and c.]

Next, we are going to try and prove the theorem twice, by turning and fitting these triangles together in different ways.

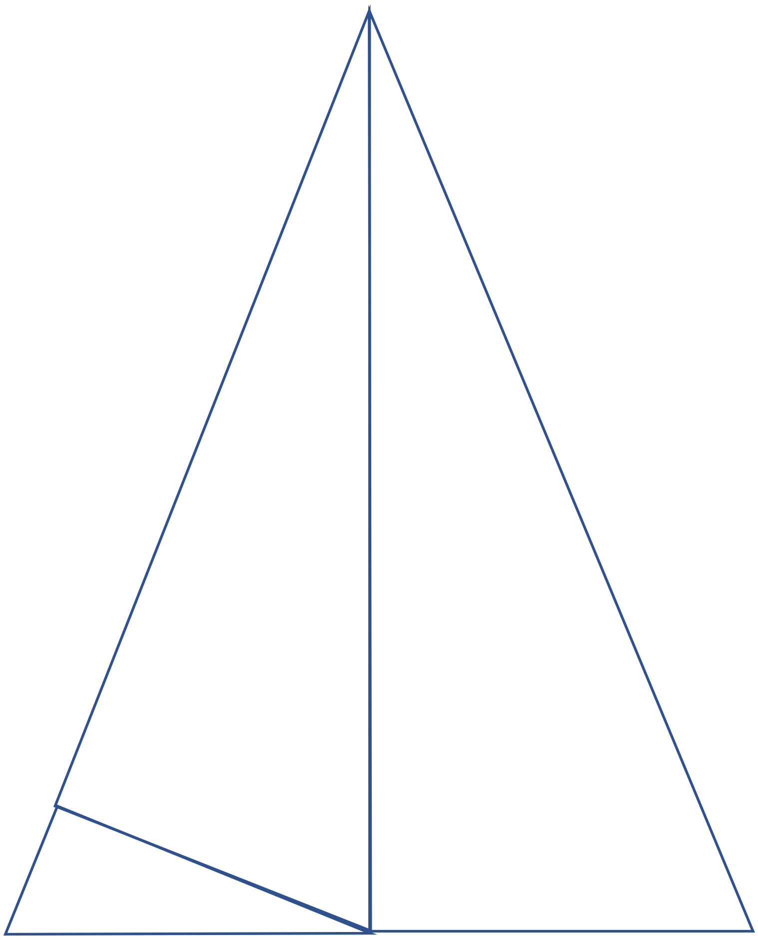
Notice that they share some side lengths in common.

Use these three triangles to make the shapes below (we can assume $a < b < c$).

(a) Explain carefully (without using the result we are trying to prove) why the diagram below shows a rectangle, and hence deduce Pythagoras' result. Refer to and label both sides and angles.



(b) Explain carefully (without using the result we are trying to prove) why the diagram below shows an isosceles triangle, and hence deduce Pythagoras' result. Refer to and label both sides and angles.



END OF QUESTIONS. CHECK YOUR WORK